COMBINATORICA
Akadémiai Kiadó – Springer-Verlag

COMBINATORICA 11 (3) (1991) 261–273

DIRECTED CIRCUITS ON A TORUS

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Received November 25, 1988
Revised August 24, 1989

Given vertices s, t of a planar digraph G, does there exist a directed circuit of G containing both s and t? We give a polynomial algorithm for this and for a number of related problems, including one about disjoint directed circuits of prescribed homotopy in a digraph drawn on a torus.

1. Introduction

Deciding if two given vertices of a graph lie in a circuit is easy, but deciding if two given vertices of a directed graph (more briefly, *digraph*) lie in a directed circuit is NP-complete [2]. (Circuits have no "repeated" vertices or edges.) For planar digraphs, however, we shall show that the problem can be solved in polynomial time. We shall also describe an "obstruction", the non-existence of which is necessary and sufficient for the existence of the required circuit. Our obstruction is rather like the familiar cutset of the max-flow min-cut theorem, with one curious difference: if we regard our graph as drawn on the sphere with the two special vertices at the north and south poles, then our obstruction involves closed curves on the sphere which wind around it several times, not just once.

Let us be more precise. By a *surface* we mean a connected (not necessarily compact) oriented 2-manifold, possibly with boundary, such as the open cylinder $\{(x,y,z) : x^2+y^2=1,\ -\infty < z < \infty\}$ or the torus (with some fixed orientation). (In fact, except for the torus, all our surfaces will be homeomorphic to subspaces of the sphere.) Since our graphs are mostly drawn on surfaces, it is convenient to make little distinction between graphs and drawings. Thus, by a *graph G in a surface Σ* we mean a pair (U,V), where

(i) $U \subseteq \Sigma$ is closed, and $V \subseteq U$
(ii) for every component e of $U-V$, $(e,\bar{e})$ is homeomorphic either to $((0,1),[0,1])$ or to $(S^1-\{1\},S^1)$.

(S^1 denotes the circle $\{e^{i\theta} : 0 \le \theta < 2\pi\}$, and for a subset e of Σ, $\bar{e}$ denotes its closure.) We permit infinite graphs, but all our graphs will be finite except when we say so. We write $U(G)=U, V(G)=V$, and define $E(G)$ to be the set of components of $U-V$ (thus, $E(G)$ is the edge set of G). A *digraph* is a graph together with an

AMS subject classification (1980): 05 C 70, 05 C 10, 05 C 20, 05 C 38

orientation of each edge of G and a *digraph in* Σ is defined similarly. We sometimes use the notation $(u,v) \in E(G)$, referring to an edge of G from u to v, even when G is in a surface. The remainder of our terminology is standard.

A *curve* (respectively, *closed curve*) in Σ is a continuous function $\phi : [0,1] \to \Sigma$ (respectively, $\phi : S^1 \to \Sigma$). We define $dom(\phi) = [0,1]$ or S^1 respectively and $\overline{\phi} = \{\phi(x) : x \in dom(\phi)\}$. If G is a graph in Σ, we say ϕ in *sensible* for G if $\{x \in dom(\phi) : \phi(x) \in U(G)\}$ is finite, and for every $x \in dom(\phi)$ and $e \in E(G)$, if $\phi(x) \in e$ then ϕ crosses e at x (in the natural sense, which it seems unnecessary to make precise). In particular, ϕ does not touch any edge "tangentially". Let z move from 0 to 1 if $dom(\phi) = [0,1]$, or let $z = e^{i\theta}$ where θ moves from 0 up to 2π if $dom(\phi) = S^1$. Let G be a digraph in Σ and let $x \in dom(\phi)$ with $\phi(x) \in e \in E(G)$. Then as z passes through x we see that $\phi(z)$ crosses e either from left to right, or from right to left (in the natural sense, determined by the orientation of Σ). For brevity, we say that ϕ crosses e at x from left to right or from right to left. If ϕ crosses e at x from left to right, we define $\alpha(x) = -1$; if it crosses from right to left, $\alpha(x) = 1$; and if $x \in dom(\phi)$ with $\phi(x) \in V(G)$, we define $\alpha(x) = 0$. If $dom(\phi) = [0,1]$ the *trace* of ϕ is the sequence $\alpha(x_1), \ldots, \alpha(x_k)$, where

$$\{x \in dom(\phi) : \phi(x) \in U(G)\} = \{x_1, \ldots, x_k\}$$

and $x_1 < x_2 < \ldots < x_k$. The *internal trace* of ϕ is the sequence $\alpha(y_1), \ldots, \alpha(y_k)$ where

$$\{x \in dom(\phi) - \{0,1\} : \phi(x) \in U(G)\} = \{y_1, \ldots, y_k\}$$

and $y_1 < y_2 \ldots < y_k$.

We want to define a similar notion for closed curves, and to do so we need "circular sequences". Thus, if $x_1, \ldots, x_k$ is a (finite) sequence and $1 \leq i \leq k$, the sequence

$$x_{i+1}, x_{i+2}, \ldots, x_k, x_1, x_2, \ldots, x_i$$

is called a *rotation* of $x_1, \ldots, x_k$. The relation "is a rotation of" is an equivalence relation, and we call its equivalence classes *circular sequences.* If $x_1, \ldots, x_k$ is a sequence, the circular sequence containing it is denoted by $\langle x_1, \ldots, x_k \rangle$. If G is a digraph in Σ and ϕ is a sensible closed curve, its *trace* is

$$\langle \alpha(e^{i\theta_1}), \ldots, \alpha(e^{i\theta_k}) \rangle$$

where

$$\{z \in S^1 : \phi(z) \in U(G)\} = \{e^{i\theta_1}, \ldots, e^{i\theta_k}\}$$

and $0 \leq \theta_1 < \theta_2 < \ldots < 2\pi$.

Let μ be a finite sequence each term of which is 1, -1 or 0. Let us choose a subsequence, and change each 0 in it to either 1 or -1. We say that μ *dominates* the new sequence of ± 1's that we obtain. The circular sequence $\langle \mu \rangle$ *dominates* $\langle \mu' \rangle$ if μ dominates some rotation of μ'. If μ is a finite sequence and $n \geq 0$ is an integer, μ^n denotes the sequence obtained by concatenating n copies of μ. Thus, if μ is $x_1, \ldots, x_k$ then μ^n is $x_1, \ldots, x_k, x_1, \ldots, x_k, \ldots, x_1, \ldots, x_k$, with nk terms. If μ_1 is a circular sequence, μ_1^n denotes $\langle \mu_2^n \rangle$ where $\mu_1 = \langle \mu_2 \rangle$.

Now any continuous function $\theta : S^1 \to S^1$ has an integral *winding number* $w(\theta)$, defined in the natural way; for instance, the identity function has winding number 1.

(Intuitively, we count the number of times $\theta(z)$ moves around S^1 as z travels around S^1 once.) Let Σ be a surface, and let $\psi : \Sigma \to S^1$ be some fixed continuous function. Then for any closed curve ϕ in Σ we define its *winding number relative to* ψ to be $w(\psi \cdot \phi)$. Finally, we can state our first result.

(1.1) *Let Σ be the sphere $\{(x,y,z) : x^2+y^2+z^2 = 1\}$, let $s = (0,0,1)$, $t = (0,0,-1)$, and for each $p = (x,y,z) \in \Sigma - \{s,t\}$ let $\psi(p) = \frac{x+iy}{(x^2+y^2)^{1/2}}$. For any digraph G in Σ with $s, t \in V(G)$, the following are equivalent:*

(i) there is a directed circuit of G containing both s and t

(ii) for every sensible closed curve ϕ in $\Sigma - \{s,t\}$, if $w(\psi \cdot \phi) = n \geq 0$ then the trace of ϕ dominates $\langle -1, 1\rangle^n$.

It is not apparent that this yields a "good characterization" for (i), because there is no bound on n. But in section 6 we shall see that (1.1) yields a polynomial algorithm for (i).

2. A generalization

Let Σ be the cylinder $\{(x,y) : 1 \leq x^2 + y^2 \leq 4\}$, oriented so that the closed curve ϕ given by $\phi(e^{i\theta}) = \left(\frac{3}{2}\cos\theta, \frac{3}{2}\sin\theta\right)$ crosses the straight line from (1, 0) to (2, 0) from right to left. Let $C_i = \{(x,y) : x^2 + y^2 = i^2\}$ $(i = 1,2)$. Let G be a digraph in Σ. By a *rung* we mean a directed path of G with one end in C_1 and the other in C_2. Let $\mathcal{P}$ be a set of vertex-disjoint rungs of G. Let $|\mathcal{P}| = k$, and choose $\theta_1 < \theta_2 < \ldots < \theta_k < \theta_1 + 2\pi$ such that for $1 \leq i \leq k$, $(\cos\theta_i, \sin\theta_i)$ is an end of some $P_i \in \mathcal{P}$. Let $\alpha_i = 1$ if $(\cos\theta_i, \sin\theta_i)$ is the first vertex of P_i, and $\alpha_i = -1$ if it is the last vertex. The circular sequence $\langle \alpha_1, \ldots, \alpha_k\rangle$ is called the *signature* of $\mathcal{P}$. We observe that it is independent of the choice of $\theta_1, \ldots, \theta_k$.

(2.1) *Let Σ be as above and for each $p = (x,y) \in \Sigma$ let $\psi(p) = \frac{x+iy}{(x^2+y^2)^{1/2}}$. Let μ be a circular sequence with all entries ± 1. Then the following are equivalent:*

(i) there is a collection of vertex-disjoint rungs of G with signature μ

(ii) for every sensible closed curve ϕ in Σ, if $w(\psi \cdot \phi) = n \geq 0$ then the trace of ϕ dominates μ^n.

This will be our second result, and we see that it is a generalization of (1.1). (To obtain (1.1) we split s, t into a number of 1-valent vertices, and we take μ to be $\langle -1, 1\rangle$. The orientation of Σ can in this case be neglected because of the invariance of μ under reversal.)

We wish also to discuss a result about disjoint circuits on a torus. This result is closely analogous to (2.1), both in statement and in proof, but unfortunately we have not been able to formulate a convenient common generalization. We therefore shall give the proof of the torus result in detail, and leave the reader to adapt it to prove (2.1). (The proof of (2.1) is in fact a little easier than that of the torus result.)

Our torus result is as follows. Let Σ be the torus obtained from the cylinder $\{(x,y,z) : x^2 + y^2 = 1, 0 \leq z \leq 1\}$ by making the identifications $(x,y,0) = (x,y,1)$.

Let λ be the closed curve given by $\lambda(e^{i\theta}) = (\cos\theta, \sin\theta, \frac{1}{2})$ $(0 \le \theta < 2\pi)$. We orient Σ so that λ crosses the curve ϕ given by

$$\phi(x) = \left(1, 0, \frac{1}{4} + \frac{1}{2}x\right) \quad (0 \le x \le 1)$$

from left to right. We define homotopy in Σ in the usual way. Let G be a digraph in Σ. A directed circuit of G is a *hoop* if it is homotopic to one of λ, λ^{-1}, where λ^{-1} is defined by $\lambda^{-1}(z) = \lambda(z^{-1})$.

Let $\mathscr{P}$ be a collection of vertex-disjoint hoops in G, and let $k = |\mathscr{P}|$. For each $P \in \mathscr{P}$, let ϕ_P be a simple closed curve (that is, $\phi_P(z) \ne \phi_P(z')$ for $z \ne z'$) with $\overline{\phi}_P = U(P)$, homotopic to λ. Then $\mathscr{P}$ may be numbered as $\{P_1, \dots, P_k\}$ such that for $1 \le i < k$, no $P \in \mathscr{P}$ lies in the open cylinder to the left of ϕ_{P_i} and to the right of $\phi_{P_{i+1}}$. We define the *signature* of $\mathscr{P}$ to be $\langle \alpha_1, \dots, \alpha_k \rangle$, where $\alpha_i = 1$ if P_i is homotopic to λ and -1 otherwise. We observe that this is independent of the choice of numbering of $\mathscr{P}$.

For each $p = (x, y, z) \in \Sigma$, let $\psi(p) = e^{2\pi i z}$. Our torus result is

(2.2) *With Σ, ψ as above, let μ be a circular sequence of ± 1's. If G is a digraph in Σ, the following are equivalent:*

(i) there is a collection of vertex-disjoint hoops of G with signature μ

(ii) for every closed curve ϕ in Σ with $w(\psi \cdot \phi) = n \ge 0$, the trace of ϕ dominates μ^n.

3. Some lemmas

The following is an easy exercise, and we omit its proof.

(3.1) *Let G be a digraph in a sphere Σ, and let r_1, r_2 be regions. Then exactly one of the following holds:*

(i) there is a directed circuit of G with r_1 on its right and r_2 on its left

(ii) there is a sensible curve ϕ in Σ such that $\phi(0) \in r_1$, $\phi(1) \in r_2$ and all its trace entries are -1.

Let Σ be the cylinder $\{(x, y, z) : x^2 + y^2 = 1,\ -\infty < z < \infty\}$ and let λ be the closed curve given by $\lambda(e^{i\theta}) = (\cos\theta, \sin\theta, 0)$. Let Σ be oriented so that λ crosses the curve ϕ given by $\phi(x) = \left(1, 0, x - \frac{1}{2}\right)$ $(0 \le x \le 1)$ from left to right. If G is a (possibly infinite) digraph in Σ, and C is a directed circuit of G, we say C is a *positive* (or *negative*, respectively) *hoop* if C is homotopic in Σ to λ (or to λ^{-1}, respectively). If C is a hoop, then $\Sigma - U(C)$ has two components Σ_1 and Σ_2 say, where Σ_1 contains every point (x, y, z) with $x^2 + y^2 = 1$ and z sufficiently large. We define $A(C) = \overline{\Sigma}_1$ and $B(C) = \overline{\Sigma}_2$; thus $A(C) \cap B(C) = U(C)$.

(3.2) *Let G be a digraph in Σ, and let C_1, C_2 be positive hoops of G such that $U(G) \subseteq A(C_1) \cap B(C_2)$. Let $v_1 \in V(C_1) - V(C_2)$, $v_2 \in V(C_2) - V(C_1)$, and suppose that every positive hoop of G contains v_1 or v_2. Then there is a sensible curve ϕ in Σ such that:*

(i) $\overline{\phi} \subseteq A(C_1) \cap B(C_2)$

(ii) $\phi(0) = v_1$, $\phi(1) = v_2$
(iii) every entry of the internal trace of ϕ *is* -1.

Proof. Let $G' = G \setminus \{v_1, v_2\}$; then G' is a digraph in Σ with no positive hoop. Let r_i be the region of G' containing v_i $(i = 1, 2)$. By (3.1) we deduce that there is a curve ϕ in Σ, sensible for G', with $\phi(0) \in r_1$ and $\phi(1) \in r_2$, such that $\overline{\phi}$ contains no vertex of G', and crosses no edge of G' from right to left. We may assume that $\overline{\phi} \cap r_1$ and $\overline{\phi} \cap r_2$ are both homeomorphic to the half-open interval $(0, 1]$ and in particular are connected, that $\phi(0) = v_1$ and $\phi(1) = v_2$, and that $\overline{\phi}$ intersects no edge of G incident with v_1 or v_2 (by "locally" rerouting ϕ). But then it follows that $\overline{\phi} \subseteq A(C_1) \cap B(C_2)$, and the theorem is satisfied. ∎

(3.3) *Let G be a (possibly infinite) digraph in Σ. Let $X \subseteq V(G)$, such that there are positive hoops C_1, C_2 of G with $V(C_1) \subseteq X$ and $V(C_2) \cap X = \emptyset$, and with $A(C_1) \cap B(C_2)$ containing only finitely many vertices and edges of G. Then there is a positive hoop C of G with $V(C) \cap X = \emptyset$, such that for every $p \in V(C)$ there is a sensible curve ϕ with $\phi(0) \in X, \phi(1) = p$ and with internal trace all -1's.*

Proof. Let us choose C_1, C_2 as in the theorem, with $A(C_1) \cap B(C_2)$ minimal. We claim that setting $C = C_2$ satisfies the theorem. For let $p \in V(C_2)$. Let C_3 be a positive hoop with $U(C_3) \subseteq A(C_1) \cap B(C_2)$ such that $p \notin V(C_3)$ and $A(C_3) \cap B(C_2)$ is minimal. From the minimality of $A(C_1) \cap B(C_2)$ it follows that $V(C_3) \cap X \neq \emptyset$; choose $q \in V(C_3) \cap X$. Let G' be the subdigraph of G drawn in $A(C_3) \cap B(C_2)$. Then G' is finite. From (3.2) applied to G', p, q the result follows. ∎

4. Periodic Digraphs

Again, let Σ be the cylinder $\{(x, y, z) : x^2 + y^2 = 1,\ -\infty < z < \infty\}$, and let G be an infinite digraph in Σ. We say that G is *periodic* if

(i) the homeomorphism $(x, y, z) \to (x, y, z + 1)$ of Σ maps G to itself,
(ii) any bounded subset of Σ contains only finitely many vertices of G, and
(iii) each vertex of G has finite valency.

Let Σ' be the torus, obtained from $\{(x, y, z) : x^2 + y^2 = 1,\ 0 \leq z \leq 1\}$ by making the identifications $(x, y, 0) = (x, y, 1)$. There is a natural surjection $\sigma : \Sigma \to \Sigma'$ defined by $\sigma(x, y, z) = (x, y, z')$, where $0 \leq z' < 1$ and $z - z'$ is an integer. The image of a periodic digraph G in Σ under σ is a finite digraph in $\overline{\Sigma}'$ which we denote by $\sigma(G)$. Conversely, every (finite) digraph in Σ' is the image under σ of some periodic digraph in Σ. This will enables us to reformulate our problem about a digraph in Σ' in terms of a periodic digraph in Σ.

If G is periodic and $v \in V(G)$, where $v = (x, y, z)$, and k is an integer, then $(x, y, z + k) \in V(G)$ and we denote this vertex by $v + (0, 0, k)$. We shall use a similar notation for the translates of other objects, without further definition.

(4.1) *Let G be a periodic digraph in Σ. Then exactly one of the following holds:*
(i) G has a positive hoop
(ii) there is a sensible curve ϕ in Σ, with trace all -1's, such that $\phi(1) = \phi(0) + (0, 0, n)$ for some integer $n > 0$.

Proof. If ϕ is a curve in Σ and $\phi(1) = \phi(0) + (0, 0, n)$ for some integer $n > 0$, and G has a positive hoop, then from the periodicity of G it follows that there is a positive hoop C with $\phi(1) \in A(C)$ and $\phi(0) \in B(C)$; and hence some entry of the trace of ϕ is 0 or 1. Thus not both (i) and (ii) hold.

For the converse, suppose that G has no positive hoop. We may assume (by perturbing G slightly) that $U(G)$ contains only finitely many points (x, y, z) with $z = 0$, and (by adding extra vertices to G) that every such point is a vertex of G. For each integer i, let

$$\Delta_i = \{(x, y, z) : x^2 + y^2 = 1, \quad i - 1 \leq z \leq i\}$$

and let G_i be the subdigraph of G drawn in Δ_i. Let N be an integer greater than the number of regions of G_1. By (3.1) applied to the finite digraph $G_1 \cup \ldots \cup G_N$, we deduce that there is a sensible curve ϕ with $\overline{\phi} \subseteq \Delta_1 \cup \ldots \cup \Delta_N$, $\phi(0) \in \Delta_0 \cap \Delta_1$, $\phi(1) \in \Delta_N \cap \Delta_{N+1}$ and with trace all -1's. Choose $0 \leq x_1 < x_2 < \ldots < x_N \leq 1$ such that $\phi(x_i) \notin U(G)$ and $\phi(x_i) \in \Delta_i - (\Delta_{i-1} \cup \Delta_{i+1})$ $(1 \leq i \leq N)$. For $1 \leq i \leq N$, let r_i be the region of G_i with $\phi(x_i) \in r_i$. Since N is greater than the number of regions of G_1, there exist i, i' with $1 \leq i < i' \leq N$ such that $r_{i'} = r_i + (0, 0, i' - i)$ in the natural sense. But then (ii) follows easily, as required. ∎

5. Proof of Theorem 2.2

Let Σ, Σ', σ be as in section 4, let G be a periodic digraph in Σ, and let μ be a non-null finite sequence of ± 1's. We wish to introduce several statements, (1), ..., (6) below, which we shall later show to be equivalent. Let $V(G) = V$.

(1) *There is a collection of vertex-disjoint hoops of $\sigma(G)$ with signature $\langle \mu \rangle$.*

For $p = (x, y, z) \in \Sigma'$, let $\psi(p) = e^{2\pi i z}$.

(2) *For every sensible closed curve ϕ in Σ', if $w(\psi \cdot \phi) = n \geq 0$ then its trace dominates $\langle \mu \rangle^n$.*

Let H be the infinite digraph with $V(H) = V$ in which u is adjacent to $v \neq u$ in H if there is a sensible curve ϕ in Σ with $\phi(0) = u$, $\phi(1) = v$ whose internal trace does not dominate μ. We define H^+ to be the infinite digraph obtained from H by adding new edges from each vertex v to $v + (0, 0, 1)$ and to $v + (0, 0, -1)$. For each $e \in E(H^+)$ we define $\ell(e) = 1$ unless $e = (u, v) \in E(H^+) - E(H)$ and $v = u + (0, 0, -1)$, when $\ell(e) = -1$.

(3) *For every directed circuit C of H^+,* $\sum_{e \in E(C)} \ell(e) \geq 0$.

(4) *There is a function $p : V \to \mathbb{Z}$ (the integers) such that if $(u, v) \in E(H)$ then $p(v) \leq p(u) + 1$, and $p(v + (0, 0, 1)) = p(v) + 1$ for all $v \in V$.*

A subset $X \subseteq V$ is *connected* if for every partition (X_1, X_2) of X with $X_1, X_2 \neq \emptyset$, there exists $v_1 \in X_1$ and $v_2 \in X_2$ incident with the same region of G.

(5) *There is a function p as in (4), such that in addition $\{v \in V : p(v) \leq j\}$ is connected for all $j \in \mathbb{Z}$.*

(6) *Let μ be $\alpha_1, \ldots, \alpha_k$; then there are k vertex-disjoint hoops $C_1, \ldots, C_k$ of G, such that $U(C_{i+1}) \subseteq A(C_i)$ $(1 \leq i < k)$ and $U(C_1 + (0,0,1)) \subseteq A(C_k)$, and $C_1 + (0,0,1)$ is vertex-disjoint from C_k, and for $1 \leq i \leq k$ C_i is a positive hoop if and only if $\alpha_i = 1$.*

We shall prove the following expanded form of (2.2).

(5.1) (1), …, (6) *are equivalent.*

It is clear that (6) $\Longrightarrow$ (1) $\Longrightarrow$ (2); and it remains to prove that (2) $\Longrightarrow$ (3) $\Longrightarrow$ (4) $\Longrightarrow$ (5) $\Longrightarrow$ (6). These implications will be shown in the following lemmas.

(5.2) (2) $\Longrightarrow$ (3).

Proof. Suppose that (2) holds, and yet some directed circuit C of H^+ fails to satisfy (3). Let the edges of $E(C) \cap E(H)$ be $e_1, \ldots, e_t$, in order. Then there are t sensible curves $\phi_1, \ldots, \phi_t$ in Σ such that

(i) for $1 \leq i \leq t$, the internal trace of ϕ_i does not dominate μ

(ii) for $1 \leq i \leq t$, $\phi_i(0) = \phi_{i-1}(1) + (0,0,-n_i)$ for some integer n_i, where ϕ_0 means ϕ_t, and

(iii) $t < n$, where $n = \sum_i n_i$.

Now the $\sigma \cdot \phi_i$'s are curves in Σ', and their concatenation (with appropriate adjustment of domain) yields a sensible closed curve ϕ say in Σ', with $w(\psi \cdot \phi) = n$. By (2), ϕ dominates $\langle \mu \rangle^n$. We deduce that the internal trace of one of $\sigma \cdot \phi_i, \ldots, \sigma \cdot \phi_t$ dominates μ, since $t < n$; and hence so does the internal trace of one of $\phi_1, \ldots, \phi_t$, a contradiction. Thus (2) $\Longrightarrow$ (3), as required. ∎

(5.3) (3) $\Longrightarrow$ (4).

Proof. Choose a vertex $v_0 \in V$. Now H (and hence H^+) is strongly connected, because any two vertices on a common region of G are joined both ways by edges of H. (It is here that we use that μ is non-null.) For any vertex $v \in V$ we define $p(v)$ as follows. Choose a directed path Q of H^+ from v to v_0. For every directed walk W from v_0 to v,

$$v_0, e_1, v_1, \ldots, e_k, v_k = v$$

say, define $\ell(W) = \sum_{1 \leq i \leq k} \ell(e_i)$. From (3), $\ell(W) + \sum_{e \in E(Q)} \ell(e) \geq 0$, and so $\ell(W)$ is bounded below. Since $\ell(W)$ is an integer, we may choose W to minimize $\ell(W)$, and we define $p(v) = \ell(W)$. Now if $e = (u, v) \in E(H^+)$, let W be a directed walk from v_0 to u with $\ell(W) = p(u)$; then extending W by e and v yields a walk from v_0 to v, and hence $\ell(W) + \ell(e) \geq p(v)$, that is, $\ell(e) \geq p(v) - p(u)$. If $e = (u, v) \in E(H)$ then $\ell(e) = 1$, and we deduce that $p(v) - p(u) \leq 1$. Moreover, for all $v \in V$, let e be the edge of H^+ from v to $v + (0,0,1)$, and f the edge from $v + (0,0,1)$ to v. Then

$$p(v + (0,0,1)) - p(v) \leq \ell(e) = 1$$
$$p(v) - p(v + (0,0,1)) \leq \ell(f) = -1,$$

that is, $p(v + (0,0,1)) = p(v) + 1$. This completes the proof. ∎

(5.4) (4) $\Longrightarrow$ (5).

Proof. If p satisfies (4), a region r of G is *p-flat* if $p(v) = 0$ for every $v \in V$ incident with r. We observe that if r is p-flat, then for each integer $n \neq 0$, the region $r + (0, 0, n)$ is not p-flat. For let $v \in V$ be incident with r (this exists, since $V \neq \emptyset$); then $v' = v + (0, 0, n)$ is incident with $r + (0, 0, n)$ and $p(v') = p(v) + n \neq 0$. It follows that the number of p-flat regions of G is at most the number of regions of $\sigma(G)$. Hence we may choose p, satisfying (4), with as many p-flat regions as possible. We shall show that p satisfies (5).

Let J be the infinite graph with $V(J) = V$, in which distinct v_1, v_2 are adjacent if they are incident with the same region of G. We claim that

(7) *For each $t \in V$ there exists $s \in V$ with $p(s) = p(t) - 1$ and a path P of J between s and t such that $p(v) \leq p(t)$ for all $v \in V(P)$.*

For suppose not. By translating, we may assume that $p(t) = 0$. Choose $X \subseteq V$ maximal such that $t \in X$, $p(v) = 0$ for all $v \in X$, and X is connected. From our assumption and the maximality of X it follows that

(8) *If $(u, v) \in E(J)$ and $u \notin X$ and $v \in X$ then $p(u) \geq 1$.*

Let $\tilde{X} = \{v + (0, 0, n) : v \in X,\ n \in \mathbf{Z}\}$. We claim that

(9) $\tilde{X} \neq V$.

For since G is periodic, there exists $(u, v) \in E(J)$ with $p(u) < p(v)$, and by translating we may assume that $p(v) = 0$. Then $u \notin X$, and so by (8), $v \notin X$. Hence $X \neq \{w \in V : p(w) = 0\}$ and so $\tilde{X} \neq V$, as claimed.

Define p' by

$$\begin{aligned} p'(v) &= p(v) + 1 \qquad && (v \in \tilde{X}) \\ &= p(v) && (v \notin \tilde{X}). \end{aligned}$$

We claim that p' satisfies (4). For suppose that there exists $(u, v) \in E(H)$ with $p'(v) \geq p'(u) + 2$. There is a sensible curve ϕ with $\phi(0) = u$ and $\phi(1) = v$, such that the internal trace of ϕ does not dominate μ. Choose u, v, ϕ such that the trace of ϕ has as few terms as possible. Since $p(v) \leq p(u) + 1$, $p(u) \leq p'(u)$, $p'(v) \leq p(v) + 1$, and $p'(v) \geq p'(u) + 2$, we have equality in all four inequalities, and in particular $v \in \tilde{X}$, $u \notin \tilde{X}$, and $p(v) = p(u) + 1$. We may assume by translating that $v \in X$, and since $u \notin X$ and $p(v) = p(u) + 1$, it follows from (8) that $(u, v) \notin E(J)$. Hence the internal trace of ϕ is non-null. It follows that there exists $v' \in V$ on a common region of G with v, and a sensible curve ϕ' with $\phi'(0) = u$, $\phi'(1) = v'$, such that the internal trace of ϕ' does not dominate μ and the trace of ϕ' has fewer terms than the trace of ϕ. From the choice of ϕ, we deduce that

$$p'(v') \leq p'(u) + 1 = p'(v) - 1 = p(v) = 0.$$

If $v' \in X$, then $0 = p(v') < p'(v') \leq 0$, a contradiction. If $v' \notin X$, then since $(v', v) \in E(J)$, it follows from (8) that $1 \leq p(v') \leq p'(v') \leq 0$, again a contradiction. Thus there is no such pair $(u, v) \in E(H)$, and so p' satisfies (4).

From the maximality of X, no p-flat region is incident with a vertex of G in $\tilde{X}$ and with a vertex in $V - \tilde{X}$; and so if r is a p-flat region, then either r or $r + (0, 0, -1)$

is p'-flat. Moreover, there is by (9) a region r of G incident both with some vertex in $\tilde{X}$ and with some vertex in $V - \tilde{X}$. Let Y be the set of vertices of G incident with r. Choose $v \in Y \cap \tilde{X}$ with $p(v)$ maximum. By translating, we may assume that $p(v) = 0$, and so $v \in X$. Now for all $u \in Y - \{v\}$

(i) $p(u) \leq p(v) + 1 = 1$, since $(u, v) \in E(H)$ (because μ is non-null)

(ii) $p(u) \geq 1$ if $u \notin X$, by (8)

(iii) $p(u) \leq p(v) = 0$ if $u \in Y \cap \tilde{X}$, by the choice of v.

It follows that $p(u) = 1$ if $u \in Y - \tilde{X}$, and $p(u) = 0$ if $u \in Y \cap \tilde{X}$. Thus $r + (0, 0, -1)$ is p'-flat, and hence there are more p'-flat regions than p-flat regions, a contradiction to the choice of p. This proves (7).

To complete the proof of (5.4) we proceed as follows. For every pair a, b of vertices of G with $p(a) = p(b) = 0$, let P_{ab} be a path of J between them. Choose $n \in \mathbb{Z}$ so that for all such a, b and all $v \in V(P_{a,b})$, $p(v) \leq n$. Let $X = \{v \in V : p(v) \leq j\}$; we must show that X is connected. From the periodicity, we may assume that $j = 0$. Let s, $t \in X$, and choose $n' \geq n, -p(s), -p(t)$. There is a path P of J with $V(P) \subseteq X$ between s and s' say, where $p(s') = -n'$, by (7). Similarly there is a path Q between t and some t' with $p(t') = -n'$. Let $s' + (0, 0, n') = a$, $t' + (0, 0, n') = b$, and let $R = P_{ab} + (0, 0, -n')$. Then R is a path of J between s' and t', and $V(R) \subseteq X$. The union of P, Q, R therefore contains a path of J within X between s and t. We deduced that X is connected, as required. ∎

(5.5) (5) $\Longrightarrow$ (6).

Proof. Choose p as in (5). We observe first that if some entry of μ is 1 then G has a positive hoop. For otherwise we could choose ϕ, n as in (4.1)(ii), we could arrange that $n \geq 2$ (by concatenating ϕ and $\phi + (0, 0, 1)$ if necessary) and that $\phi(0), \phi(1) \in V$ (since $V \neq \emptyset$). But then $(\phi(0), \phi(1)) \in E(H)$ and yet $p(\phi(1)) - p(\phi(0)) = n \geq 2$, contrary to (5). Similarly, if some entry of μ is -1 then G has a negative hoop.

Let $X_1 = \{v \in V : p(v) \leq 0\}$. Since G is periodic, there are hoops C, C' of G, positive if and only if $\alpha_1 = 1$, such that $V(C) \subseteq X_1$ and $V(C') \cap X_1 = \emptyset$. By (3.3), there is a hoop C_1, positive if and only if $\alpha_1 = 1$, such that $V(C_1) \cap X_1 = \emptyset$ and for every $v \in V(C_1)$ there is a sensible curve ϕ_1 with $\phi_1(0) \in X_1$, $\phi_1(1) = v$ and with internal trace all $-\alpha_1$'s. Since X_1 is connected and $X_1 \cap B(C_1) \neq \emptyset$ it follows that $X_1 \subseteq B(C_1)$.

Inductively, having defined $C_1, \ldots, C_{i-1}$ for some i with $2 \leq i \leq k$, we define C_i as follows. Let $X_i = B(C_{i-1}) \cap V$. From the choice of C_{i-1}, it follows that X_1, $V(C_1), \ldots, V(C_{i-1}) \subseteq X_i$. By (3.3) there is a hoop C_i, positive if and only if $\alpha_i = 1$, such that $V(C_i) \cap X_i = \emptyset$ and for every $v \in V(C_i)$ there is a sensible curve ϕ_i with $\phi_i(0) \in X_i$, $\phi_i(1) = v$ and with internal trace all $-\alpha_i$'s. Since $\overline{\phi}_i$ meets $U(C_{i-1})$ and $V(C_{i-1}) \subseteq X_i$, we may assume that $\phi_i(0) \in V(C_{i-1})$. Since X_i is connected, it follows that $X_i \subseteq B(C_i)$. This completes the inductive definition.

We observe that for each $v \in V(C_k)$ there are k sensible curves $\phi_1, \ldots, \phi_k$ such that $\phi_1(0) \in X_1$, $\phi_i(0) = \phi_{i-1}(1)$ $(2 \leq i \leq k)$ and $\phi_k(1) = v$, and such that for $1 \leq i \leq k$ the internal trace of ϕ_i is all $-\alpha_i$'s. It follows that the internal trace of the concatenation of $\phi_1, \ldots, \phi_k$ does not dominate μ, and hence $(\phi_1(0), v) \in E(H)$. Since $p(\phi_1(0)) \leq 0$ we deduce that $p(v) \leq 1$, from (5). Thus $V(C_k + (0, 0, -1)) \subseteq X_1$, and so $U(C_k + (0, 0, -1)) \subseteq B(C_1)$, that is, $U(C_1 + (0, 0, 1)) \subseteq A(C_k)$. But $C_1 + (0, 0, 1)$ and C_k are disjoint since $p(v) \leq 1$ for all $v \in V(C_k)$ and $p(v) \geq 2$ for all $v \in V(C_1 + (0, 0, 1))$. Thus (6) holds. ∎ This completes the proof of (5.1). ∎

6. Algorithms

In this section Σ is the torus as in (2.2), and G is a digraph drawn on it (and λ, ψ are as before (2.2)). Our result (5.1) may be used to give a polynomial algorithm to test whether (2.2)(i) holds, because as we shall see statement (3) of section 5 can be checked in polynomial time. To do so, we first observe that we can confine ourselves to input graphs G with "2-cell" drawings, that is, in which every region is homeomorphic to an open disc. For if a region r is not an open disc then one of three possibilities holds:

(i) There is a simple closed curve in r bounding a closed disc $\Delta \subseteq \Sigma$, and $U(G) \cap \Delta \neq \emptyset$. Then the portion of G drawn in Δ may be deleted without changing the problem.

(ii) There is a simple closed curve ϕ in r, non-null-homotopic in Σ and with $w(\psi \cdot \phi) \neq 0$; then (if μ is non-null) (2.2)(ii) is false.

(iii) There is a simple closed curve ϕ in r, homotopic to λ; then we may "cut" Σ along $\overline{\phi}$, and reduce to a new problem on a cylinder, to solve which there is an easy "greedy" algorithm.

Thus we assume henceforth that G is a 2-cell drawing.

Let G^* be a dual digraph of G (defined as usual, with one vertex for each region of G; every edge of G^* crosses the corresponding edge of G from left to right). Let G^+ be a digraph in Σ with $V(G^+) = V(G) \cup V(G^*)$, with edges the edges of G^* together with, for each $w \in V(G^*)$ lying in a region r of G and for each $v \in V(G)$ with $v \in \overline{r}$, two edges, both in r, one from v to w and one from w to v.

To describe an algorithm for (2.2), we must discuss how the drawing of G is presented, as an input for the algorithm. However, we wish to be vague on this point. We shall assume merely that we are presented with G and G^+ as abstract (non-embedded) digraph, and that we have a subroutine which for each circuit C of G^+ and integer k, will decide whether $w(C) \leq k$ in time polynomial in $|V(G)| + |E(G)| + k$. (It is convenient to equip circuits with orientations, even if they are not directed circuits; thus, from now on, there are two circuits corresponding to the same "circuit subgraph" of G, with opposite senses. If C is a circuit of G, we denote by $w(C)$ the winding number of C, that is, $w(\psi \cdot \phi)$ where ϕ is a simple closed curve with $\overline{\phi} = U(C)$ tracing $U(C)$ in the same sense as C, and ψ is as in (2.2).)

Our algorithm proceeds as follows. Intuitively, we proceed to compute the graph H and function ℓ of statement (3) in section 5. However, to avoid working with infinite graphs we formulate the procedure in terms of the finite graph on the torus (which was called $\sigma(G)$ in section 5, and is now called G) and a corresponding auxiliary graph.

Step 1. Choose a spanning tree T of G^+.

(This is possible since G^+ is connected.)

For each edge $e \in E(G^+) - E(T)$, let C_e be the circuit of G^+ with $E(C_e) \subseteq E(T) \cup \{e\}$, using e in its positive sense, and let $n(e) = w(C_e)$. For each $e \in E(T)$, let $n(e) = 0$.

Step 2. For each $e \in E(G^+) - E(T)$, check that $|n(e)| \leq |V(C_e)|$.

(If not then (2.2)(ii) (and hence (2.2)(i)) is false, assuming that μ is non-null.)

Step 3. For each $e \in E(G^+)$, compute $n(e)$.

(We can do so in polynomial time, since by step 2 we may assume that $|n(e)|$ is polynomially bounded.)

We observe that for every circuit C of G^+,

$$w(C) = \sum_{e \in E(C)^+} n(e) - \sum_{e \in E(C)^-} n(e)$$

where $E(C)^+$, $E(C)^-$ are the sets of edges of C whose directions agree and disagree respectively with the sense of C.

Step 4. Check that for each directed circuit C of G^*, if some term of μ is 1 then $w(C) \leq 0$ and if some term of μ is -1 then $w(C) \geq 0$.

(This can be done in polynomial time by [1], using the numbers $n(e)$ computed in step 3 and the observation above. If some C does not have the property of step 4, then (2.2)(ii)(and hence (2.2)(i)) is false.)

If W is a walk $v_0, e_1, v_1, \ldots, e_k, v_k$ in G^+, we define

$$n(W) = \sum_{1 \leq i \leq k} \pm n(e_i)$$

taking the $+$ sign if W uses e_i in its positive direction.

Step 5. For each pair u, v of vertices of G such that there is a directed path of G^+ from u to v with no other vertex in $V(G)$:

(i) if some term of μ is 1, choose such a path $W^1_{u,v}$ with $n(W^1_{u,v})$ maximum

(ii) if some term of μ is -1, choose such a path W say with $n(W)$ minimum, and let $W^{-1}_{v,u}$ be the reverse of W.

(From step 4, these can be computed in polynomial time.)

Let μ be $\alpha_1, \ldots, \alpha_k$. Let H be the digraph with $V(H) = V(G)$, where $(u, v) \in E(H)$ if there is a sequence $u = v_0, v_1, \ldots, v_k = v$ of vertices of G such that $W^{\alpha_i}_{v_{i-1}, v_i}$ exists $(1 \leq i \leq k)$. For each $e = (u, v) \in E(H)$, let $\ell(e)$ be the maximum over all such choices of $v_0, v_1, \ldots, v_k$ of $\sum_{1 \leq i \leq k} n\left(W^{\alpha_i}_{v_{i-1}, v_i}\right)$.

Step 6. Compute the digraph H and the numbers $\ell(e)$.

(We may assume that $k \leq |V(G)|$ since otherwise (2.2)(i) is false, and so step 6 can be carried out in polynomial time.)

Step 7. Check that for every directed circuit C of H, $\sum_{e \in E(C)} \ell(e) \leq |E(C)|$.

(Again, this can be done by the method of [1].) If every circuit C has the property of step 7 then (2.2)(i) is true because of (5.1), and otherwise (2.2)(i) is false. This completes the algorithm.

We leave the proof of correctness to the reader.

7. Extension and applications

We have already said that (2.1) can be proved by a very similar proof, which we omit, and yet we have been unable to formulate a convenient common generalization of (2.1) and (2.2). Nevertheless, (2.2) does have an application to a problem about a digraph drawn on a cylinder, as follows. Let Σ be $\{(x,y) : 1 \leq x^2 + y^2 \leq 4\}$ and for $1 \leq i \leq k$, let one of s_i, t_i be $\left(\cos \frac{2\pi i}{k}, \sin \frac{2\pi i}{k}\right)$ and the other be twice that. Let G be a digraph in Σ with

$$U(G) \cap bd(\Sigma) = \{s_1, t_1, \ldots, s_k, t_k\} \subseteq V(G).$$

We ask: do there exist k vertex-disjoint directed paths $P_1, \ldots, P_k$ of G such that for $1 \leq i \leq k$, P_i is from s_i to t_i and is homotopic in Σ to the straight line from s_i to t_i? To solve this, we add an edge from t_i to s_i for each i, obtaining a digraph on a torus in the natural way, and we apply (2.2). It turns out to be necessary and sufficient that for every closed curve ϕ in Σ and for every curve ϕ in Σ with both ends in $bd(\Sigma)$, the trace of ϕ dominates the appropriate sequence. This results extends to digraphs theorem (5.10) of [3].

There are several possible generalizations of (2.2). For instance, given a digraph G on a torus Σ and λ as before, where every vertex v has an integer capacity $c(v) \geq 0$, and a sequence of ± 1's, we can ask: do there exist directed closed walks $W_1, \ldots, W_k$ in G, each homotopic to λ or to λ^{-1}, non-self-crossing and pairwise non-crossing, with signature $\langle \mu \rangle$, such that for each vertex v, $\sum_i n(W_i, v) \leq c(v)$, where $n(W_i, v)$ is the number of occurrences of v in W_i? It turns out that a result like (2.2) holds. (We modify the definition of trace so that when ϕ passes through a vertex v, its trace acquires not one but $c(v)$ 0's.) The proof is like that for (2.2).

A second curious generalization is the following. Let G, Σ etc. be as before and let μ be a sequence of ± 1's with k terms. Let $G_1, \ldots, G_k$ be subdigraphs of G. When is there a collection of vertex-disjoint hoops $\{C_1, \ldots, C_k\}$ numbered in order, so that C_i is a circuit of $G_i (1 \leq i \leq k)$ and is positive if and only if $\alpha_i = 1$? Again, a result like (2.2) holds and the proof is similar.

A third generalization would be to replace the torus by another surface. Since this paper was written, A. Schrijver has solved the corresponding problem (by a different method) for every connected surface except the torus and Klein bottle. The Klein bottle problem remains open.

Acknowledgments

This research was done during a recent conference on "Paths, flows and VLSI-design" (Bonn, 1988) and several of the participants helped with the proof, particularly András Frank, László Lovász, Neil Robertson, Lex Schrijver, Éva Tardos and Richard Vitray.

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